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2005 J. Phys. A: Math. Gen. 38 L783

(http://iopscience.iop.org/0305-4470/38/46/L03)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **38** (2005) L783–L789 doi:10.1088/0305-4470/38/46/L03

LETTER TO THE EDITOR

The stability of the replica-symmetric state in finite-dimensional spin glasses

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Received 26 September 2005
Published 2 November 2005
Online at stacks.iop.org/JPhysA/38/L783

Abstract
According to the droplet picture of spin glasses, the low-temperature phase of spin glasses should be replica symmetric. However, analysis of the stability of this state suggested that it was unstable and this instability lends support to the Parisi replica symmetry breaking picture of spin glasses. The finite-size scaling functions in the critical region of spin glasses below T_c in dimensions greater than 6 can be determined and for them the replica-symmetric solution is unstable order by order in perturbation theory. Nevertheless the exact solution can be shown to be replica symmetric. It is suggested that a similar mechanism might apply in the low-temperature phase of spin glasses in less than six dimensions, but that a replica symmetry broken state might exist in more than six dimensions.

PACS numbers: 75.10.Nr, 75.50.Lk

An important development in statistical physics was Parisi's replica symmetry breaking (RSB) solution of the Sherrington–Kirkpatrick (SK) model of spin glasses [1]. The SK model is the model for which mean-field theory is exact, by virtue of the infinite range of the interactions between the spins. The Parisi solution is characterized by a large number of pure states, organized into an ultrametric hierarchy [2]. While there is no debate as to whether the RSB solution is correct for the SK model, almost from the beginning there has been a controversy as to whether it extends to real finite-dimensional spin glasses. To extend it to finite dimensions, one constructs the loop (perturbative) expansion around the mean-field solution. At least in high dimensions this perturbative expansion is well defined i.e. all the integrals are finite. This programme is entirely conventional and most of our understanding of phase transitions and the low-temperature phase across condensed matter physics has followed this route (mean-field theory, then the fluctuations around the mean-field). How then could it fail for spin glasses? This is the question which we will attempt to answer in this letter.

The chief rival to the Parisi replica symmetry breaking (RSB) picture is the droplet picture [3]. On this picture there are just two pure states (the analogue of the up and down states of the ferromagnet), and so the low-temperature phase is replica symmetric. The properties of the spin glass phase are governed by its excitations; the free-energy cost of flipping the spins in a region of linear extent L is supposed to be L^{θ}, where the exponent $\theta \sim 0.2$ in three dimensions, according to numerical studies. The droplet has fractal area L^{d_s} where $d-1 \leqslant d_s \leqslant d$ and d is the dimensionality of the system. The droplet has a fractal surface as it wanders to pass the domain wall separating reversed spins through as many weak bonds as possible, in order to minimize its overall energy. Thus the droplet picture focuses on the low-cost excitations, which are determined by unusual correlations of the bonds, and attributes the properties of the low-temperature phase to these 'Griffiths' singularities.

The droplets are very intimately connected to non-average values of the bonds between the spins on nearby sites. However, in the replica trick the first step is to average out the bond disorder. Thus after using the replica trick, features which can be associated with droplet behaviour can only arise from a treatment which goes beyond straightforward perturbation theory. In the example which we shall discuss in detail below, at each finite order in the expansion about the replica-symmetric solution there are instabilities which might lead one to believe that replica symmetry must be broken. However, the exact solution remains replica symmetric. It is natural to wonder whether a similar mechanism might apply more generally and we shall give arguments as to why this might happen.

The replica field theory of spin glasses (see [4] for a review) starts from the Hamiltonian density

$$\mathcal{H} = \frac{1}{4}\sum_{\alpha,\beta}(\nabla q_{\alpha\beta})^2 + \frac{\tau}{4}\sum_{\alpha,\beta} q_{\alpha\beta}^2 - \frac{w}{6}\sum_{\alpha,\beta,\gamma} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha} - y\left(\frac{1}{12}\sum_{\alpha,\beta} q_{\alpha\beta}^4 + \frac{1}{8}\sum_{\alpha,\beta,\gamma,\delta} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\delta}q_{\delta\alpha} - \frac{1}{4}\sum_{\alpha,\beta,\gamma} q_{\alpha\beta}^2 q_{\alpha\gamma}^2\right). \quad (1)$$

The field components $q_{\alpha\beta}$ ($\alpha \neq \beta$ and $q_{\alpha\beta} = q_{\beta\alpha}$) take all real values, and the indices such as α take the values $1, 2, 3, \ldots, n$. In the limit when n goes to zero, it is hoped that such a field theory captures the physics of finite-dimensional spin glasses. The quartic terms are for $d < 6$ irrelevant variables [5], as they are in the finite-size critical regime in any dimension so they will be dropped. The coefficient of the quadratic term τ changes the sign at the mean-field transition temperature T_{c0} so $\tau \sim (T - T_{c0})/T_{c0}$.

We will start by briefly reviewing the kind of calculation which has led to the view that the replica-symmetric solution is unstable [6]. One begins with the mean-field solution which is a stationary point of the Hamiltonian density of equation (1). The standard replica-symmetric solution (on dropping the quartic terms) is the spatially uniform $q_{\alpha\beta}(\mathbf{x}) = Q$ independent of α and β, where $Q = \tau/(n-2)w$. Only the trivial solution $Q = 0$ exists on the high-temperature side of the transition where $\tau > 0$. The stability of this replica-symmetric solution for negative τ at and beyond mean-field theory will occupy most of this letter.

The first step is to write

$$q_{\alpha\beta} = Q + R_{\alpha\beta} \quad (2)$$

and substitute into equation (1) (without the quartic terms). Then up to constants

$$\mathcal{H}\{R_{\alpha\beta}\} = \frac{1}{2}\tau\sum_{\alpha<\beta} R_{\alpha\beta}^2 + \frac{1}{2}\sum_{\alpha<\beta}(\nabla R_{\alpha\beta})^2 - wQ \times \sum_{\alpha<\beta<\gamma}(R_{\alpha\beta}R_{\alpha\gamma} + R_{\alpha\beta}R_{\beta\gamma} + R_{\alpha\gamma}R_{\beta\gamma}) - w\sum_{\alpha<\beta<\gamma} R_{\alpha\beta}R_{\alpha\gamma}R_{\beta\gamma}. \quad (3)$$

The quadratic terms are not diagonal. It is useful to first introduce the following propagators in terms of the Fourier components $R_{\alpha\beta}(\mathbf{q})$

$$\begin{aligned} G_1(q) &= \langle R_{\alpha\beta}(\mathbf{q}) R_{\alpha\beta}(-\mathbf{q})\rangle, \\ G_2(q) &= \langle R_{\alpha\beta}(\mathbf{q}) R_{\alpha\gamma}(-\mathbf{q})\rangle, \qquad \beta \neq \gamma \\ G_3(q) &= \langle R_{\alpha\beta}(\mathbf{q}) R_{\gamma\delta}(-\mathbf{q})\rangle, \qquad \alpha, \beta \neq \gamma, \delta. \end{aligned} \tag{4}$$

Then, following [6] the quadratic form is readily diagonalized in terms of three linear combinations of G_1, G_2 and G_3:

$$\begin{aligned} G_B &\equiv G_1 + 2(n-2)G_2 + \tfrac{1}{2}(n-2)(n-3)G_3 = (q^2 + |\tau|)^{-1} \\ G_A &\equiv G_1 + (n-4)G_2 - (n-3)G_3 = (q^2 + 2wQ)^{-1} \\ G_R &\equiv G_1 - 2G_2 + G_3 = (q^2 + nwQ)^{-1}. \end{aligned} \tag{5}$$

All three of these propagators are of the form $\left(q^2 + m_s^2\right)^{-1}$, with the mass of the longitudinal mode given by $m_L^2 = |\tau|$, of the 'anomalous' mode by $m_A^2 = 2wQ$ and of the replicon mode by $m_R^2 = nwQ$. In the limit of $n \to 0$ the breather and the anomalous masses become equal while the replicon mass goes to zero. Stability of course requires that all the m_s^2 be non-negative. Thus at Gaussian order the replica-symmetric solution has marginal stability. (If we had retained the quartic terms in the Hamiltonian density the replicon mode would have become unstable at Gaussian order). To see the apparent instability of the replica-symmetric state it is necessary to go to one-loop order and calculate the self-energies of the propagators. The replicon self-energy $\Sigma_R(q)$ is defined via

$$G_R = (q^2 + nwQ - \Sigma_R(q))^{-1}. \tag{6}$$

To one-loop order the calculation of $\Sigma_R(q)$ is straightforward [6, 7],

$$\begin{aligned} \Sigma_R(0) = {} & \frac{n^4 - 8n^3 + 19n^2 - 4n - 16}{(n-1)(n-2)^2} I_{RR} + \frac{8(n-1)(n-4)}{n(n-2)^2} I_{RA} \\ & + \frac{8}{n(n-1)} I_{RL} + \frac{(n-4)^2}{(n-2)^2} I_{AA}, \end{aligned} \tag{7}$$

where for the wavevector sums we have introduced the notation

$$I_{ss'} = \frac{w^2}{N} \sum_{\mathbf{q}} \frac{1}{q^2 + m_s^2} \frac{1}{q^2 + m_{s'}^2} \tag{8}$$

and s and s' correspond to one of the subspaces R, A, L. In the limit $n \to 0$, $\Sigma_R(0)$ can be approximated as [6]

$$\Sigma_R(0) \approx \frac{4w^2|\tau|^2}{N} \sum_{\mathbf{q}} \frac{1}{(q^2 + nwQ)^2 (q^2 + |\tau|)^2}. \tag{9}$$

In the large-N limit, the sum over the wavevectors $\mathbf{q}$ in equation (9) can be converted to an integral. For $d > 8$ the integrals will exist if cutoff at $q = \Lambda$, where $\Lambda \sim 1/a$ and a is the lattice spacing. Then $\Sigma_R(0) \sim |\tau|^2$ on setting n to zero. For $4 < d < 8$, in the same limit, $\Sigma_R(0)$ does not require an upper cutoff and $\Sigma_R(0) \sim |\tau|^{(d-4)/2}$. When $d < 4$, the integral is dominated by its small q behaviour and is only finite if the replicon mass is kept finite (i.e. by not setting n to zero):

$$\Sigma_R(0) \sim \frac{1}{|nwQ|^{(4-d)/2}}. \tag{10}$$

For all dimensions to this order $\Sigma_R(0)$ is positive, implying that m_R^2 is negative and that at one-loop order the replica-symmetric state is unstable. However, in an earlier paper we argued that such a conclusion was premature [8]. One expects that deep within the ordered phase, that is, when $|\tau| \to \infty$, loop corrections should be small. However, as noted above, below four dimensions $\Sigma_R(0)$ actually becomes infinite as n goes to zero. Worse divergences exist in higher-loop corrections, as diagrams exist which diverge whenever $d < 6$ and n goes to zero. We suggested therefore that it might be incorrect to conclude that replica symmetry had to be broken for dimensions $d < 6$, since the higher the loop correction, the more divergent the diagram and that one had to go beyond order-by-order perturbation theory in dimension $d < 6$ in order to discover whether the replica-symmetric state was really unstable [8].

Additional arguments were also given in [8] as to why six might be a special dimension for the nature of the low-temperature spin glass state. If it is indeed replica symmetric below six dimensions then there should be no Almeida–Thouless (AT) line [9] as the AT line marks the onset of replica symmetry breaking. This is consistent with the renormalization group calculation of Bray and Roberts [10] who were unable to locate a stable fixed point in $6 - \epsilon$ dimensions and as a consequence speculated that this could be due to the fact that there was no AT line in these dimensions. (Simulations of three-dimensional spin glasses seem consistent with this conclusion [11], as do experiments [12].)

For dimensions $d > 6$, the loop expansion about the RSB *mean-field state* is well behaved [4] and one might therefore hope that the Parisi RSB picture for these dimensions is a valid description of the spin glass state. Below six dimensions, the loop expansion has divergences which were related to the divergences associated with non-mean-field critical exponents [4]. The loop expansion is meant to be used well away from the critical region, so it is hard to understand this identification. It is possible that they indicate instead that the RSB state is unstable in less than six dimensions.

Our old argument, that calculations to finite order in the loop expansion might not predict the correct stability of the replica-symmetric state when coefficients in the loop expansion become infinite as n goes to zero [8], met with little interest, perhaps because no concrete realization could be given. I can now supply an example where, order by order in the loop expansion, the coefficients diverge as n goes to zero, have signs which indicate that the replica-symmetric state is apparently unstable, but which can be exactly resummed to a solution which shows that the replica-symmetric state is stable, which parallels what is being argued might happen in spin glasses for $d < 6$.

This illustrative calculation is of the low-temperature form of the finite-size critical scaling functions for $d > 6$, which can be done 'exactly'. It is as follows. One supposes that the spin glass system is of finite linear extent L and that the system has periodic boundary conditions. $N = L^d$. Because of the periodicity of the system, the order parameter will be still uniform. One can proceed to construct a loop expansion as for the infinite system, and in expressions such as equation (7) the sum over wavevector components such as q_x runs over values $0, \pm 2\pi/L, \pm 4\pi/L, \ldots$. The Fourier components of $q_{\alpha\beta}(\mathbf{q})$ at nonzero values of $\mathbf{q}$ are in this context 'massive' modes and can be traced out perturbatively. Only the $\mathbf{q} = 0$ component has to be treated non-perturbatively. Above the upper critical dimension the arguments of Brézin and Zinn-Justin [13] show that the effect of integrating out the nonzero $\mathbf{q}$ modes is simply to shift the mean-field transition temperature to the true transition and to renormalize the value of the coupling constant w. The properties of the $\mathbf{q} = 0$ fields, such as those in Binder ratio plots studied in simulations, can then all be extracted from the partition function without the gradient terms

$$Z = \int \prod_{\alpha<\beta} \left(\frac{\mathrm{d}q_{\alpha\beta}}{\sqrt{2\pi}} \right) \exp \left[-\frac{L^d \tau}{4} \sum_{\alpha,\beta} q_{\alpha\beta}^2 + \frac{L^d w}{6} \sum_{\alpha,\beta,\gamma} q_{\alpha\beta} q_{\beta\gamma} q_{\gamma\alpha} \right]. \tag{11}$$

A typical ratio would be

$$M_6 = \frac{\langle (Trq^3)^2 \rangle}{(\langle Trq^2 \rangle)^3}. \tag{12}$$

where the thermal averaging denoted by the angular brackets is done using the partition function of equation (11). By rescaling the $q_{\alpha\beta}$ fields it is easy to see that $M_6 = g(N\tau^3/w^2)$. The scaling functions such as $g(x)$ are the quantities of interest and can be calculated from suitable derivatives of the partition function Z. The neglect of higher terms such as the quartic terms of equation (1) is easily justified in the finite-size critical scaling region where $N \to \infty$, $\tau \to 0$, but with $N\tau^3/w^2$ finite.

The partition function Z has recently been extensively studied in [14]. For earlier studies see [15]. On the high-temperature side of the transition i.e. when $\tau > 0$, we showed that the series expansion in the variable $w^2/N\tau^3$, while formally a divergent series, looks as though it could be resummed to give useful results. The technique adopted to study the high-order terms in the perturbation expansion was to map the problem onto the spherical Ising spin glass in the Sherrington–Kirkpatrick (SK) limit [16], Our concern in this letter is what happens on the low-temperature side when $\tau < 0$.

Let us examine the first term in equation (7) for finite N. In sums over $\mathbf{q}$, the $\mathbf{q} = 0$ terms coming from the replicon propagators are infinite unless we keep n finite since the replicon propagator at $\mathbf{q} = 0$ is just $1/nwQ$. Thus the most divergent part in equation (7) is of order $w^2/N(nwQ)^2$ and arises from the term I_{RR}. On the other hand, the sums over nonzero q values can be approximated by the expressions previously given for $\Sigma_R(0)$. Both types of terms make $\Sigma_R(0)$ positive and apparently indicate that expanding about the replica-symmetric state would lead to the usual instabilities. However, the apparent divergence of terms like $w^2/N(nwQ)^2$ as $n \to 0$ has first to be resolved.

Such divergences occur in all orders of the expansion and the most divergent terms come when all the propagators are of replicon type. At Kth order the general form of these contributions from $\mathbf{q} = 0$ in the replicon sector is $(w^2/N(nwQ)^2)^K$ and retaining these terms alone gives terms of the schematic form

$$\begin{aligned} G_R(0)^{-1} &\sim nwQ - \frac{w^2}{N(nwQ)^2} + \frac{w^4}{N^2(nwQ)^5} + \cdots \\ &= nwQf(w^2/N(nwQ)^3). \end{aligned} \tag{13}$$

Thus if the function $f(x)$ has the large-x form, $x^{\frac{1}{3}}$, $G_R(0) \sim N^{\frac{1}{3}} / w^{\frac{2}{3}}$ and the problematic n dependence would disappear. The argument below shows that this is indeed what happens. It arises from the mapping to the spherical spin glass in the SK limit [14]. $G_R(0)$ physically is equal to

$$G_R(0) = \frac{1}{N} \sum_{i,j} [\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle]^2. \tag{14}$$

$G_R(0)$ can be studied directly in the spherical spin glass in the low-temperature regime. It can be approximated using the procedures in [14] by

$$G_R(0) = \frac{1}{N} \sum_{\lambda}{}' \frac{1}{\lambda^2}, \tag{15}$$

where the eigenvalues λ are related to those of a random symmetric J_{ij} matrix, such that in the large-N limit

$$\rho(\lambda) = \frac{N}{2\pi}[\lambda(4-\lambda)]^{\frac{1}{2}}, \tag{16}$$

and the prime in the sum indicates that the smallest (or negative eigenvalues) are to be omitted from the sum in the finite-N case. Using the infinite-N form for the density of states the integral for $G_R(0)$ would appear to be divergent from the behaviour of the integrand at small λ. However, the integral can be cutoff at the smallest eigenvalue, which is of order $1/N^{2/3}$, when one can see that $G_R(0)$ is indeed of order $N^{1/3}$. In principle by doing numerical work on the finite-N spherical model, it would be possible to calculate $G_R(0)$ in the finite-size scaling limit to any desired accuracy.

Returning now to the loop calculation, the problematic terms which became infinite as n goes to zero have been resummed so that now

$$G_R(0)^{-1} \sim w^{\frac{2}{3}}\Big/N^{\frac{1}{3}} - \tau^2. \tag{17}$$

The second term of order τ^2 comes from the nonzero $\mathbf{q}$ contributions in the one-loop calculation (the quartic terms in equation (1) also give terms of order τ^2)). Note that in the finite-size scaling region $N \to \infty$, $\tau \to 0$, but with $N\tau^3$ fixed, the term of order τ^2 is of order $1/N^{2/3}$ and so is negligible in comparison to the first term at large N. The fact that it is negligible is what would have been expected anyway from the arguments of [13], who showed that the only role of the nonzero $\mathbf{q}$ contributions was to renormalize the mean-field transition temperature and coupling constant w. Note that the two terms on the right-hand side of equation (17) become comparable when $N\tau^6$ is $O(1)$. In the SK limit, the number of metastable (TAP) states N_s goes like $\ln N_s \sim N\tau^6$ for small τ [17] so that $G_R(0)^{-1}$ only goes negative indicating an instability towards replica symmetry breaking when a multiplicity of states exist—a result entirely in accord with the natural expectation.

Thus by going beyond perturbation theory we have tamed divergences in such a way that a calculation which seemingly required replica symmetry to be broken to cure an instability present at one-loop order is no longer unstable when the divergences are summed. The divergences in the finite size calculation, when resummed, give in the large-N limit a negligible contribution if one is not working in the finite-size scaling limit. However, it is my belief that the divergences which plague the loop expansion in the low-temperature state for dimensions $d < 6$ as n goes to zero require a similar non-perturbative treatment, and if this could be done, the replica-symmetric state would emerge as stable.

Very recently numerical evidence [18] has emerged that is consistent with our proposal that replica symmetry breaking occurs only when $d > 6$. The one-dimensional long-range Ising spin glass model with interactions whose magnitude decreases as $1\big/r_{ij}^{\sigma}$ was studied. When $1/2 < \sigma \leqslant 2/3$, the system is expected to behave like the short-range spin glass model with $d > 6$ and an AT line was found in this interval. No AT line was found when $2/3 < \sigma \leqslant 1$, the interval which is expected to mirror the short-range spin glass below six dimensions. These results are also consistent with earlier analytical calculations [19].

Over many years Newman and Stein have proved a number of rigorous theorems concerning the nature of the ordered phase in finite-dimensional spin glasses [20]. For $d < 6$ our proposal that the ordered phase is replica-symmetric and droplet-like is completely consistent with their theorems. For $d > 6$ their 'chaotic pairs' picture fits with our proposal (but there are also other possibilities). In the chaotic pair picture the domain wall exponent $\theta > 0$ but its fractal dimension $d_s = d$. In other words the domain walls are space filling. Also their theorems do not rule out an AT line when $d_s = d$. The *global* Parisi overlap function $P(q)$, which is closely related to the spatially uniform order-parameter $q_{\alpha\beta}$, could be non-trivial for the chaotic pair state, possibly even RSB-like. They emphasize in their work though that the global $P(q)$ is not useful for understanding the properties of pure states in finite-dimensional glasses.

Acknowledgments

I should like to acknowledge the hospitality of the Aspen Center of Physics, CO, USA during the writing of this letter. I am also very grateful to Dan Stein for patiently explaining his results with Newman to me.

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